High School Math Contest
University of South Carolina
January 26, 2013

1. Oil is pumped into a non-empty tank at a changing rate. The volume of oil in the tank doubles every minute and the tank is filled in 10 minutes. How many minutes did it take for the tank to be half full?

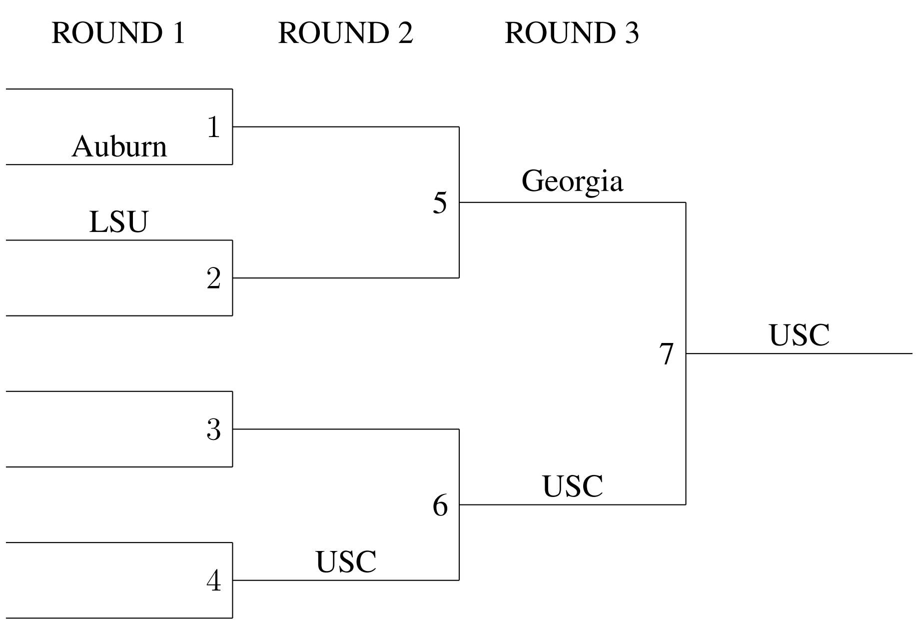
   (a) 2 (b) 5 (c) 7 (d) 8 (e) 9

2. Alice and Bill are walking in opposite directions along the same route between A and B. Alice is going from A to B, and Bill from B to A. They start at the same time. They pass each other 3 hours later. Alice arrives at B 2.5 hours before Bill arrives at A. How many hours does it take for Bill to go from B to A?

   (a) 6 (b) 6.5 (c) 7 (d) 7.5 (e) 8.5

3. A possible single-elimination tournament bracket for eight teams looks like the one below. The winner of a game advances to the next round and the winner's name gets put on the line segment that is directly connected to the slot for the game. (For example, as shown, USC wins the Slot 4 game and then gets put on the lower line segment in Slot 6.) The teams are USC, Florida, Georgia, LSU, Alabama, Auburn, Vanderbilt, and Ole Miss.

   In how many ways can the rest of the bracket be completed? (The home team for each game is the one that is listed further up, so listing team X above team Y in a slot counts as different from listing team Y above team X in the same slot.)



   (a) 64 (b) 128 (c) 187 (d) 384 (e) 768

4. You have two boxes. Each of them has a square base and is half as tall as it is wide.

   If the larger box is two inches wider than the smaller box, and has a volume 244 in^3 greater, what is the width of the smaller box?

   (a) 4 in (b) 6 in (c) 8 in (d) 10 in (e) 12 in

5. A machine was programmed to transmit a certain sequence of five digits, all zeros and ones, five times. One time it did it correctly; one time it did so with one mistake; one time it did so with two mistakes; one time it did so with three mistakes; one time it did so with four mistakes.

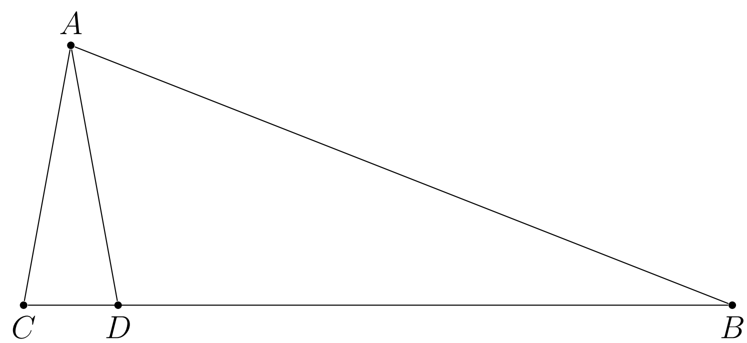
   The five transmissions are listed below. Which is the correct sequence?

   (a) 00001 (b) 00100 (c) 01100 (d) 10010 (e) 10011

6. What is the value of $\sin\left(\frac{2\pi}{5}\right) + \sin\left(\frac{4\pi}{5}\right) + \sin\left(\frac{6\pi}{5}\right) + \sin\left(\frac{8\pi}{5}\right)$?

   (a) -1 (b) $-\frac{1}{\sqrt{5}}$ (c) 0 (d) $\frac{1}{\sqrt{5}}$ (e) 1

7. In the triangle $\triangle ABC$, the point D lies on side $\overline{BC}$. Also, $AC = 3, AD = 3, BD = 8,$ and $CD = 1$. What is AB ?



   (a) 8 (b) 8.25 (c) 9 (d) 9.25 (e) 10

8. Let $f(x, y)$ be defined by (1) $f(x, 0) = x$ and (2) $f(x, y + 1) = f(f(x, y), y)$. Which of the following is the largest?

   (a) $f(10, 15)$ (b) $f(11, 14)$ (c) $f(12, 13)$ (d) $f(13, 12)$ (e) $f(14, 11)$

9. Two armies are advancing towards each other, each one at 1 mph.

   A messenger leaves the first army when the two armies are 10 miles apart and runs towards the second at 9 mph. Upon reaching the second army, he immediately turns around and runs towards the first army at 9 mph.

   How many miles apart are the two armies when the messenger gets back to the first army?

   (a) 5.6 (b) 5.8 (c) 6 (d) 6.2 (e) 6.4

10. What is the largest integer k for which $85!$ is divisible by 42^k?

    (a) 2 (b) 13 (c) 41 (d) 81 (e) 135

11. You play on a game show where a prize has been randomly put into one of five boxes, labeled A, B, C, D, and E, with each box equally likely to contain the prize.

    The boxes are closed when you first see them, and you guess that box A contains the prize. Two of the empty boxes from among the four other boxes are then opened.

    If these two boxes are C and E, what is the probability that box D contains the prize?

    (a) $\frac{1}{5}$ (b) $\frac{1}{3}$ (c) $\frac{2}{5}$ (d) $\frac{3}{5}$ (e) $\frac{2}{3}$

12. Consider the set of all solutions (x, y) to the equation $3x^2 = 3y^4 + 2013$, where x and y are integers.

    What is the sum of all values of y occurring in such pairs (x, y) ?

    (a) 0 (b) 5 (c) 36 (d) 41 (e) 1296

13. Six mountain climbers divide themselves into three teams for the final assault on a peak. One team has 3 climbers; the others have 1 and 2 climbers.

    All manners of different team deployments are considered, which team goes first, second, and third, and two deployments are considered different if the composition of any of the teams is different. (Disregard deployments within each team.)

    What is the total number of possible team deployments?

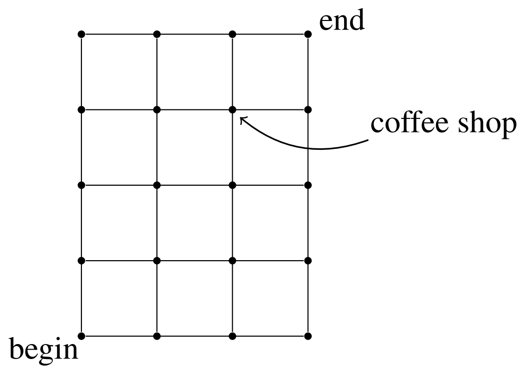
    (a) 60 (b) 180 (c) 360 (d) 720 (e) none of the above

14. How many positive integers less than or equal to 2013 are divisible by at least one of 3, 11, and 61?

    (a) 3 (b) 813 (c) 1006 (d) 1198 (e) 2013

15. A man wants to walk 4 blocks north and 3 blocks east. At each step he goes north or east. When there is an option, he is equally likely to go in either direction.

    One block southwest of his destination, there is a coffee shop at the corner. What is the probability that he passes this corner on either side of the street?



(a) $2/5$ (b) $3/7$ (c) $1/2$ (d) $4/7$ (e) $3/5$

16. What is the number of solutions of the equation $\sin(x) = \dfrac{x}{3140}$?

    (The real number x is in radians.)

(a) 0 (b) 628 (c) 1999 (d) 2013 (e) infinite

17. You play on a game show with the following rules. You are shown five prizes with different prices. You are given five price tags which give the prices of the prizes, but you don't know which tag goes with which prize.

    You are asked to match the price tags to the prizes. Unfortunately, you have no idea how much any of the prizes costs, and so you place the price tags randomly.

    What is the probability you place exactly two price tags correctly?

(a) $\dfrac{1}{6}$ (b) $\dfrac{1}{5}$ (c) $\dfrac{3}{10}$ (d) $\dfrac{2}{5}$ (e) $\dfrac{1}{2}$

18. If $f(x) = x^{10} + 2x^9 - 2x^8 - 2x^7 + x^6 + 3x^2 + 6x + 1$, then what is $f(\sqrt{2} - 1)$?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

19. Among all collections of distinct positive integers whose sum is 20, let n be the largest possible product. What is the sum of the digits of n ?

(a) 2 (b) 9 (c) 16 (d) 18 (e) 23

20. If the minimum value of the function $f(x) = \cos(2x) - 2a(1 + \cos(x))$ is $-\frac{1}{2}$, then what is the value of a?

(a) $-2 - \sqrt{3}$ (b) $-1 - \sqrt{3}$ (c) -2 (d) $-2 + \sqrt{3}$ (e) $2 - \sqrt{3}$

21. There are 480 flies sitting on the squares of a 16×30 board, with one fly in each square. We say that two flies are neighbors if their squares share a side. They take off, and then land on a 15×32 board, again with one fly in each square. What is the probability that each fly has every neighbor the same on both boards?

(a) 0 (b) $\frac{480}{480} \cdot \frac{4}{479} \cdot \frac{3}{478} \cdot \frac{2}{477} \cdot \frac{1}{476} \cdot \frac{475}{475} \cdot \frac{4}{474} \cdot \frac{3}{473} \cdot \frac{2}{472} \cdot \frac{1}{471} \cdot \ldots \cdot \frac{5}{5} \cdot \frac{4}{4} \cdot \ldots \cdot \frac{1}{1}$

(c) $\frac{4}{479} \cdot \frac{4}{478} \cdots \frac{4}{1}$ (d) 1 (e) none of the above

22. The lengths of the edges of three cubes are integers when measured in centimeters. If the sum of their surface areas is 564 cm^2, which of the following can be the sum of their volumes?

(a) Only 564 cm^3 (b) Either 564 cm^3 or 586 cm^3 (c) Only 586 cm^3

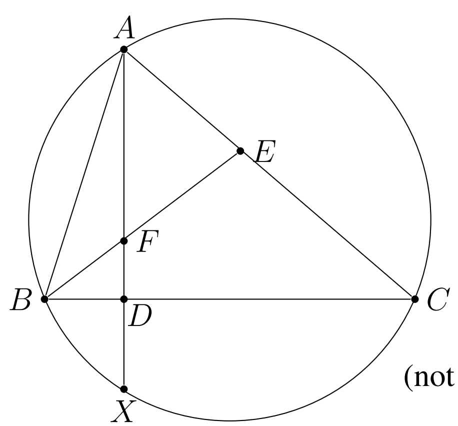
(d) Either 586 cm^3 or 764 cm^3 (e) Only 764 cm^3

23. For what values of k does the equation $\log_{10}(kx) = 2\log_{10}(x + 1)$ have only one real root? (Here, $\log_{10}$ refers to the logarithm to the base 10.)

(a) $k < -1$ or $k = 2$ (b) $k < 0$ or $k = 2$ (c) $k < 1$ or $k = 2$ (d) $k < 0$ or $k = 4$

(e) $k < 0$ or $k = 8$

24. Triangle $\triangle ABC$ with sides $AB = 6$, $BC = 7$, and $AC = 8$ is inscribed in a circle as shown. The point F is on the altitude $\overline{AD}$ and $FD = DX$. Find $\sin(\angle BEA)$.



(not drawn accurately)

(a) $\frac{6}{7}$ (b) $\frac{3}{4}$ (c) $\frac{7}{8}$ (d) $\frac{4\sqrt{3}}{7}$ (e) 1

25. Let F_n denote the Fibonacci numbers, defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$.

Let S be the sum

$$\sum_{n=1}^{10} \frac{F_n}{5^n} = \frac{F_1}{5} + \frac{F_2}{5^2} + \cdots + \frac{F_{10}}{5^{10}} \quad .$$

Which of the following numbers is closest to S?

(a) $\frac{5}{19}$ (b) $\frac{4}{19}$ (c) $\frac{1}{19}$ (d) $\frac{1}{4}$ (e) $\frac{1}{2}$

26. Let x and y be real numbers. What is the minimum value of the following function

$$f(x, y) = \sqrt{4 + y^2} + \sqrt{(x-2)^2 + (2-y)^2} + \sqrt{(4-x)^2 + 1} \ ?$$

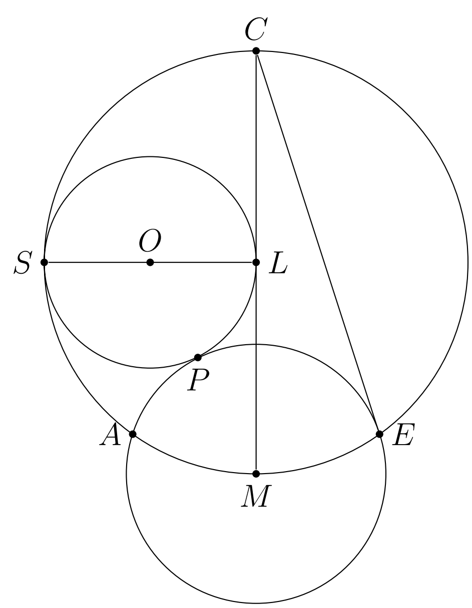
(a) 0 (b) $\frac{7}{2}$ (c) 5 (d) 25 (e) 49

27. Three circles are situated as follows.

The circle SPL has diameter $\overline{SL}$ and it is centered at O; the circle APE is centered at M; and the circle CAE has diameter $\overline{CM}$ and it is centered at L.

The circle CAE is tangent to the circle SPL at S and the circles SPL and APE are tangent at P. Also, $\overline{OL} \perp \overline{CM}$.

Find the angle $\angle MCE$.



(a) 15° (b) 18° (c) 20° (d) 24° (e) 30°

28. How many different quadruples (x_1, x_2, x_3, x_4) satisfy the equation $x_1 + x_2 + x_3 + x_4 = 18$, where each x_i is a positive integer ≤ 8 ?

(a) 83 (b) 172 (c) 256 (d) 344 (e) 352

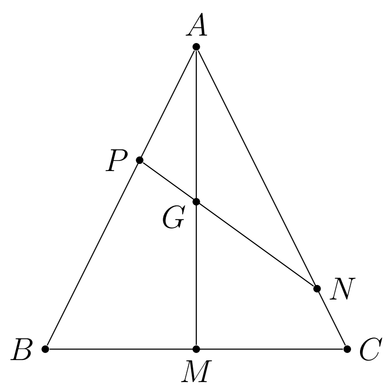
29. Let x, y, and z be positive real numbers which satisfy the following equations:

$$x^2 + xy + y^2 = 3$$
$$y^2 + yz + z^2 = 1$$
$$z^2 + zx + x^2 = 4.$$

What is the value of the expression $xy + yz + zx$?

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{2}}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) 2 (e) $\frac{2}{\sqrt{3}}$

30. In the triangle $\triangle ABC$, the points P and N are on the sides $\overline{AB}$ and $\overline{CA}$, respectively. Also, $\frac{AP}{PB} = \frac{1}{2}$ and $\frac{CN}{NA} = \frac{1}{4}$. What is the ratio $\frac{AG}{GM}$ in which the segment $\overline{PN}$ divides the median $\overline{AM}$?



(a) $\frac{1}{2}$ (b) $\frac{5}{7}$ (c) $\frac{7}{8}$ (d) $\frac{8}{9}$ (e) $\frac{8}{7}$