High School Math Contest
University of South Carolina
January 26, 2013
Solutions

1. Oil is pumped into a non-empty tank at a changing rate. The volume of oil in the tank doubles every minute and the tank is filled in 10 minutes. How many minutes did it take for the tank to be half full?

   (a) 2 (b) 5 (c) 7 (d) 8 (e) 9

   Answer: (e)

   Solution: At the 10 minute mark, the tank has twice as much oil in it as it did at the 9 minute mark. So the tank became half-full at the 9 minute mark.

2. Alice and Bill are walking in opposite directions along the same route between A and B. Alice is going from A to B, and Bill from B to A. They start at the same time. They pass each other 3 hours later. Alice arrives at B 2.5 hours before Bill arrives at A. How many hours does it take for Bill to go from B to A?

   (a) 6 (b) 6.5 (c) 7 (d) 7.5 (e) 8.5

   Answer: (d)

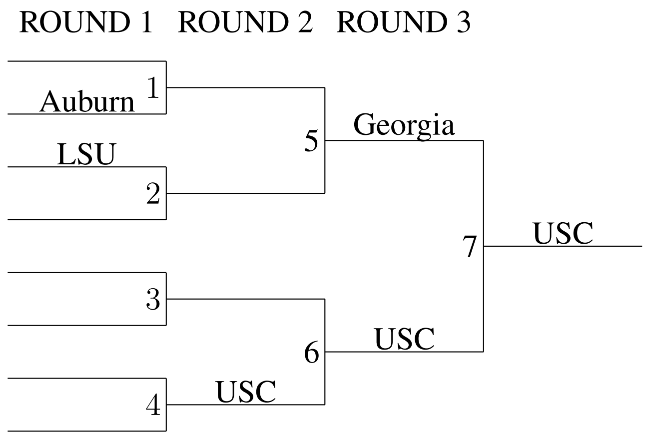
   Solution: Suppose that it takes Alice x hours to go from A to B. Then, it takes Bill $x + 2.5$ hours to go from B to A. Assuming that Alice and Bill maintain constant speed, we get that Alice's speed is d/x and Bill's speed is $d/(x + 2.5)$, where d is the distance from A to B. Since Alice and Bill met 3 hours after they started walking,

   $$3\left(\frac{d}{x} + \frac{d}{x + 2.5}\right) = d.$$

   Dividing by d and solving for x we get $x = 5$, so the answer is $5 + 2.5 = 7.5$ hours.

3. A possible single-elimination tournament bracket for eight teams looks like the one below. The winner of a game advances to the next round and the winner's name gets put on the line segment that is directly connected to the slot for the game. (For example, as shown, USC wins the Slot 4 game and then gets put on the lower line segment in Slot 6.) The teams are USC, Florida, Georgia, LSU, Alabama, Auburn, Vanderbilt, and Ole Miss.

   In how many ways can the rest of the bracket be completed? (The home team for each game is the one that is listed further up, so listing team X above team Y in a slot counts as different from listing team Y above team X in the same slot.)



(a) 64 (b) 128 (c) 187 (d) 384 (e) 768

Answer: (d)

Solution: There are two slots that Georgia could have originated from, 1 or 2. There are four choices available for the other slot. Georgia is on the upper line of slot 5, and there two choices for the other slot. So there are $2 \cdot 4 \cdot 2 = 16$ possible outcomes resulting in Georgia advancing to Round 3.

On the lower half of the diagram, there are two choices for USC's Round 1 placement, and there are $3! = 6$ ways to fill the other three Round line segments, and two different outcomes for the slot 3 game, giving $2 \cdot 6 \cdot 2$ possible outcomes resulting in USC advancing to Round 3. So the total number of ways the bracket can be completed is $16 \cdot 24 = 384$.

4. You have two boxes. Each of them has a square base and is half as tall as it is wide.

   If the larger box is two inches wider than the smaller box, and has a volume 244 in^3 greater, what is the width of the smaller box?

   (a) 4 in (b) 6 in (c) 8 in (d) 10 in (e) 12 in

   Answer: (c)

   Solution: If w is the width of the smaller box, then the volume of this box is $w^3/2$, and the volume of the larger box is $(w+2)^3/2$. We get $\dfrac{(w+2)^3}{2} - \dfrac{w^3}{2} = 244$. Multiplying by 2, expanding $(w+2)^3$, and simplifying we get $w^2 + 2w = 80$. Since w is positive, we get $w = 8$.

5. A machine was programmed to transmit a certain sequence of five digits, all zeros and ones, five times. One time it did it correctly; one time it did so with one mistake; one time it did so with two mistakes; one time it did so with three mistakes; one time it did so with four mistakes.

   The five transmissions are listed below. Which is the correct sequence?

   (a) 00001 (b) 00100 (c) 01100 (d) 10010 (e) 10011

   Answer: (b)

Solution: Let us call ”distance” between transmissions the number of positions in which they differ. We need to find which of the five transmissions has distances to the other four transmissions, 1, 2, 3, and 4 in some order. Now, the distances from (a) to the other transmissions are 2,3,2,3; from (b): 2,1,3,4; from (c): 3,1,4,5; from (d): 3,3,4,1; from (e): 2,4,5,1. Thus, the answer is (b).

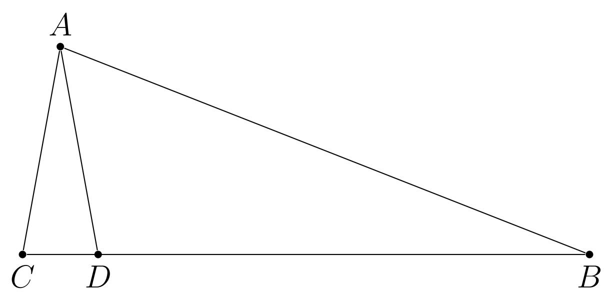
6. What is the value of $\sin\left(\frac{2\pi}{5}\right)+\sin\left(\frac{4\pi}{5}\right)+\sin\left(\frac{6\pi}{5}\right)+\sin\left(\frac{8\pi}{5}\right)$?

(a) -1 (b) $-\frac{1}{\sqrt{5}}$ (c) 0 (d) $\frac{1}{\sqrt{5}}$ (e) 1

Answer: (c)

Solution: Since $\sin(2\pi-x)=\sin(-x)=-\sin(x)$ for all real numbers x, we have $\sin\left(\frac{8\pi}{5}\right)=-\sin\left(\frac{2\pi}{5}\right)$, and $\sin\left(\frac{6\pi}{5}\right)=-\sin\left(\frac{4\pi}{5}\right)$. Therefore the sum is 0.

7. In the triangle $\triangle ABC$, the point D lies on side $\overline{BC}$. Also, $AC=3$, $AD=3$, $BD=8$, and $CD=1$. What is AB ?



(a) 8 (b) 8.25 (c) 9 (d) 9.25 (e) 10

Answer: (c)

Solution: The triangles $\triangle ABC$ and $\triangle DAC$ have a common angle $\angle C$. Also, $\frac{BC}{AC}=\frac{BD+DC}{AC}=\frac{8+1}{3}=3$ and $\frac{AC}{DC}=\frac{3}{1}=3$. Thus, the two triangles are similar. It follows that $\frac{AB}{DA}=3$, and then $AB=9$.

8. Let $f(x,y)$ be defined by (1) $f(x,0)=x$ and (2) $f(x,y+1)=f(f(x,y),y)$. Which of the following is the largest?

(a) $f(10,15)$ (b) $f(11,14)$ (c) $f(12,13)$ (d) $f(13,12)$ (e) $f(14,11)$

Answer: (e)

Solution: $f(x,1)=f(f(x,0),0)=f(x,0)=x, f(x,2)=f(f(x,1),1)=f(x,1)=x,\cdots,f(x,n)=x$. So the biggest value is $f(14,11)$, since it has the largest x value.

9. Two armies are advancing towards each other, each one at 1 mph.

   A messenger leaves the first army when the two armies are 10 miles apart and runs towards the second at 9 mph. Upon reaching the second army, he immediately turns around and runs towards the first army at 9 mph.

   How many miles apart are the two armies when the messenger gets back to the first army?

   (a) 5.6 (b) 5.8 (c) 6 (d) 6.2 (e) 6.4

   Answer: (e)

   Solution: The messenger reaches the second army in one hour (the messenger and the second army advance at each other at a combined speed of 10 mph). At that time, the two armies are 8 miles apart. It takes 0.8 hours for the messenger to get back to the first army. At that time the armies are $8 - 0.8 - 0.8 = 6.4$ miles apart.

10. What is the largest integer k for which $85!$ is divisible by 42^k?

    (a) 2 (b) 13 (c) 41 (d) 81 (e) 135

    Answer: (b)

    Solution: Note that $42 = 2 \cdot 3 \cdot 7$. There are 12 multiples of 7 from 1 to 85. These are $7, 14, 21, \ldots, 84$. There is one multiple of $7^2 = 49$, too. So, the exponent of 7 in the prime factorizations of $85!$ is 13. It is easy to see that exponents of 2 and 3 are lager than 13. So, the largest k with the desired property is 13.

11. You play on a game show where a prize has been randomly put into one of five boxes, labeled A, B, C, D, and E, with each box equally likely to contain the prize.

    The boxes are closed when you first see them, and you guess that box A contains the prize. Two of the empty boxes from among the four other boxes are then opened.

    If these two boxes are C and E, what is the probability that box D contains the prize?

    (a) $\frac{1}{5}$ (b) $\frac{1}{3}$ (c) $\frac{2}{5}$ (d) $\frac{3}{5}$ (e) $\frac{2}{3}$

    Answer: (c)

    Solution: The probability of the prize being in box A is equal to that of it being in any other box; hence this probability is $1/5$. The remaining $4/5$ is split between Box D and Box B; hence the probability that Box D contains the prize is $2/5$.

12. Consider the set of all solutions (x, y) to the equation $3x^2 = 3y^4 + 2013$, where x and y are integers.

    What is the sum of all values of y occurring in such pairs (x, y) ?

    (a) 0 (b) 5 (c) 36 (d) 41 (e) 1296

    Answer: (a)

    Solution: Note that if (x, y) is one solution, so is $(x, -y)$. Thus the sum of the y values is 0.

Remark: One can show that the equation has at least one integer solution and only finitely many solutions. Indeed, the equations can be rewritten as $(x - y^2)(x + y^2) = 671$. So, if x and y is an integer solution with x and y positive, then $x + y^2$ divides 671, so $|x|$ and y^2 can not be larger than 671. Also, setting $x + y^2 = 61$, $x - y^2 = 11$ leads to the solution $x = 36$, $y = 5$.

13. Six mountain climbers divide themselves into three teams for the final assault on a peak. One team has 3 climbers; the others have 1 and 2 climbers.

    All manners of different team deployments are considered, which team goes first, second, and third, and two deployments are considered different if the composition of any of the teams is different. (Disregard deployments within each team.)

    What is the total number of possible team deployments?

    (a) 60 (b) 180 (c) 360 (d) 720 (e) none of the above

    Answer: (c)

    Solution: There are six ways to choose which team goes first, which goes second and which goes third. Also, there are $\binom{6}{3} = 20$ ways to select three climbers out of the six for the three man team, and 3 ways to pick a climber out of the remaining three for the one man team. So, the total number of ways is $6 \cdot 20 \cdot 3 = 360$ ways.

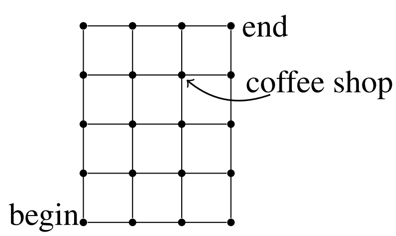
14. How many positive integers less than or equal to 2013 are divisible by at least one of 3, 11, and 61?

    (a) 3 (b) 813 (c) 1006 (d) 1198 (e) 2013

    Answer: (b)

    Solution: There are $2013/3 = 671$ multiples of 3 from 1 to 2013, $2013/11 = 183$ multiples of 11, and $2013/61 = 33$ multiples of 61. To avoid double counting we subtract the 61 multiples of $3 \cdot 11$, the 11 multiples of $3 \cdot 61$, and the 3 multiples of $61 \cdot 11$. We have to add 1 for 2013 which was subtracted one too many times. Thus, the answer is $671 + 183 + 33 - 61 - 11 - 3 + 1 = 813$.

15. A man wants to walk 4 blocks north and 3 blocks east. At each step he goes north or east. When there is an option, he is equally likely to go in either direction. One block southwest of his destination, there is a coffee shop at the corner. What is the probability that he passes this corner on either side of the street?



    (a) 2/5 (b) 3/7 (c) 1/2 (d) 4/7 (e) 3/5

Answer: None of the above.

Remark: This problem was not counted when determining the results of the contest. The answer is $\frac{5}{16}$.

Solution: Let us introduce coordinate system with origin at "begin". For x and y nonnegative integers let $F(x, y)$ be the probability that the man reaches (x, y) along his way to $(3, 4)$. Clearly, $F(0, 0) = 1$. Moreover, $F(1, 0) = F(0, 1) = \frac{1}{2}$ and $F(2, 0) = F(0, 2) = \frac{1}{4}$. Also, for $0 < x < 3$ and $0 < y < 4$, we have $F(x + 1, y + 1) = \frac{1}{2}F(x + 1, y) + \frac{1}{2}F(x, y + 1)$ (we can get to $(x + 1, y + 1)$ only via $(x + 1, y)$ or $(x, y + 1)$). Using this formula repeatedly, we get $F(2, 3) = \frac{5}{16}$.

16. What is the number of solutions of the equation $\sin(x) = \dfrac{x}{3140}$?

    (The real number x is in radians.)

    (a) 0 (b) 628 (c) 1999 (d) 2013 (e) infinite

    Answer: (c)

    Solution: Note that the equation contains two solutions in each of the intervals $[2k\pi, (2k+1)\pi]$ for all $k = 0, \ldots, 499$ (the range of $\sin x$ in each of these intervals is $[0, 1]$ and $0 < x/3140 < 1$ in these intervals, since $999\pi < 999 \cdot 3.142 < 3140$). Thus, there are 999 positive solutions, 0 is a solution, and there are 999 negative solutions (both $\sin(x)$ and x are odd functions, so x is a solution only if $-x$ is a solution). Clearly, there are no solutions for $x > 1000\pi > 3140$, since $x/3140 > 1$ for such x.

17. You play on a game show with the following rules. You are shown five prizes with different prices. You are given five price tags which give the prices of the prizes, but you don't know which tag goes with which prize.

    You are asked to match the price tags to the prizes. Unfortunately, you have no idea how much any of the prizes costs, and so you place the price tags randomly.

    What is the probability you place exactly two price tags correctly?

    (a) $\frac{1}{6}$ (b) $\frac{1}{5}$ (c) $\frac{3}{10}$ (d) $\frac{2}{5}$ (e) $\frac{1}{2}$

    Answer: (a)

    Solution: Suppose that the prizes are A, B, C, D, and E. There are $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to place all the price tags. First of all, observe that there are $\frac{5 \cdot 4}{2} = 10$ pairs of prizes which you might have gotten right. Suppose that you get A and B right. Then there are two ways to get the other three prizes wrong: you can give C the tag for D, D the tag for E, and E the tag for C – or you can do the reverse of this. For each of the 10 pairs of prizes, there are similarly two ways to get the prices wrong, so there are a total of $10 \cdot 2 = 20$ ways to get exactly two prizes correct. $\frac{20}{120} = \frac{1}{6}$.

18. If $f(x) = x^{10} + 2x^9 - 2x^8 - 2x^7 + x^6 + 3x^2 + 6x + 1$, then what is $f(\sqrt{2} - 1)$?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

Answer: (d)

Solution: Note that $a = \sqrt{2} - 1$, is a root of $g(x) = (x+1-\sqrt{2})(x+1+\sqrt{2}) = x^2 + 2x - 1$. Using long division we get $f(x) = g(x)(x^8 - x^6 + 3) + 4$. Thus, $f(a) = g(a)(a^8 - a^6 + 3) + 4 = 0 + 4 = 4$.

19. Among all collections of distinct positive integers whose sum is 20, let n be the largest possible product. What is the sum of the digits of n ?

(a) 2 (b) 9 (c) 16 (d) 18 (e) 23

Answer: (b)

Solution: We claim that the largest possible product is $2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$, so the answer is 9.

First of all, note that it is pointless to include 1. Assuming we exclude 1, we have room for at most 5 integers, and there are no other products with at least 5 integers.

Next, note that we cannot have a gap larger than 2 between the integers we pick. Indeed, if k and $k+l$ are two consecutive integers with $l \geq 3$, then we can replace them by $k+1$, $lk+l-1$, because $k(k+l) = k^2 + kl < k^2 + kl + (l-1) = (k+1)(k+l-1)$.

Moreover, there can be at most a gap of size 2, since $k, k+2$, and l, $l+2$ with $k+2 \leq l$ are four of the numbers we picked, then we can replace k, $l+2$ by $k+1$, $l+1$, because $k(l+2) = kl + 2k < kl + k + l + 1 = (k+1)(l+1)$.

The only other products left to consider are $3 \cdot 4 \cdot 6 \cdot 7 = 504$, $5 \cdot 7 \cdot 8 = 280$, and $9 \cdot 11 = 99$. All these products are less than 720, so the largest product is 720.

20. If the minimum value of the function $f(x) = \cos(2x) - 2a\,(1 + \cos(x))$ is $-\dfrac{1}{2}$, then what is the value of a?

(a) $-2 - \sqrt{3}$ (b) $-1 - \sqrt{3}$ (c) -2 (d) $-2 + \sqrt{3}$ (e) $2 - \sqrt{3}$

Answer: (d)

Solution: We rewrite $f(x) = 2\cos^2 x - 1 - 2a - 2a\cos x = 2\left(\cos x - \dfrac{a}{2}\right)^2 - \dfrac{a^2}{2} - 2a - 1$, and consider the following possible cases:

$Case1$: If $a > 2$, then $f(x)$ takes the minimum $1 - 4a$ when $\cos x = 1$.

$Case2$: If $a < -2$, then $f(x)$ takes the minimum 1 when $\cos x = -1$.

$Case3$: If $-2 \leq a \leq 2$, then $f(x)$ takes the minimum $-\dfrac{a^2}{2} - 2a - 1$ when $\cos x = \dfrac{a}{2}$.

In the first two cases the minimum of $f(x)$ cannot be $-\dfrac{1}{2}$.

Therefore, we have $-\dfrac{a^2}{2} - 2a - 1 = -\dfrac{1}{2}$. The roots are $a = -2 + \sqrt{3}$, or $a = -2 - \sqrt{3}$ and since a is in $[-2, 2]$, we have $a = -2 + \sqrt{3}$.

21. There are 480 flies sitting on the squares of a 16×30 board, with one fly in each square. We say that two flies are neighbors if their squares share a side. They take off, and then land on a 15×32 board, again with one fly in each square. What is the probability that each fly has every neighbor the same on both boards?

(a) 0 (b) $\frac{480}{480} \cdot \frac{4}{479} \cdot \frac{3}{478} \cdot \frac{2}{477} \cdot \frac{1}{476} \cdot \frac{475}{475} \cdot \frac{4}{474} \cdot \frac{3}{473} \cdot \frac{2}{472} \cdot \frac{1}{471} \cdot \ldots \cdot \frac{5}{5} \cdot \frac{4}{4} \cdot \ldots \cdot \frac{1}{1}$

(c) $\frac{4}{479} \cdot \frac{4}{478} \cdots \frac{4}{1}$ (d) 1 (e) none of the above

Answer: (a)

Solution: Since $14 \cdot 28 = 392$ of the flies had four neighbors on the old board, but only $13 \cdot 30 = 390$ have four neighbors on the new board, the answer is probability 0.

22. The lengths of the edges of three cubes are integers when measured in centimeters. If the sum of their surface areas is 564 cm^2, which of the following can be the sum of their volumes?

(a) Only 564 cm^3 (b) Either 564 cm^3 or 586 cm^3 (c) Only 586 cm^3

(d) Either 586 cm^3 or 764 cm^3 (e) Only 764 cm^3

Answer: (d)

Solution: Suppose the side lengths of three cubes are a, b, c. We have $6(a^2 + b^2 + c^2) = 564$, $\Rightarrow a^2 + b^2 + c^2 = 94$. Without loss of generality, suppose $1 \leq a \leq b \leq c < 10$, so $3c^2 \geq a^2 + b^2 + c^2 = 94$, $c^2 > 31$. Therefore, $6 \leq c < 10$. So c can only be $9, 8, 7, 6$.

If $c = 9$, $\Rightarrow a^2 + b^2 = 94 - 9^2 = 13$, we have $a = 2, b = 3$ one solution $(a, b, c) = (2, 3, 9)$.

If $c = 8$, $\Rightarrow a^2 + b^2 = 94 - 8^2 = 30$, we have $b = 4$ or $b = 5$, but a has no integer solution.

If $c = 7$, $\Rightarrow a^2 + b^2 = 94 - 7^2 = 45$, we have $a = 3, b = 6$ one solution $(a, b, c) = (3, 6, 7)$.

If $c = 6$, $\Rightarrow a^2 + b^2 = 94 - 6^2 = 58$, we have $2b^2 \geq a^2 + b^2 = 58$, $b^2 \geq 29$, so $b = 6$, but a has no integer solution.

Then the volume is $V_1 = 2^3 + 3^3 + 9^3 = 764$ or $V_2 = 3^3 + 6^3 + 7^3 = 586$.

23. For what values of k does the equation $\log_{10}(kx) = 2\log_{10}(x + 1)$ have only one real root? (Here, $\log_{10}$ refers to the logarithm to the base 10.)

(a) $k < -1$ or $k = 2$ (b) $k < 0$ or $k = 2$ (c) $k < 1$ or $k = 2$ (d) $k < 0$ or $k = 4$

(e) $k < 0$ or $k = 8$

Answer: (d)

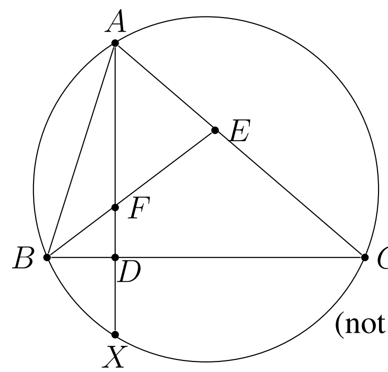
Solution: Note that we need $kx > 0$ and $x + 1$ so that the log functions are defined. We consider two cases.

Case 1. $k < 0$. Here, we need that the equation (*) $(x + 1)^2 = kx$ has exactly one solution x between -1 and 0. Since on the interval $[-1, 0]$ the function $(x + 1)^2$ increases from 0 to to 1, and the function kx decreases from $-k$ to 0, the equation (*) has exactly one solution in $(-1, 0)$, so all values of $k < 0$ work.

Case 2. $k > 0$. Here we need that the equation (*) has exactly one positive solution. If x_1 and x_2 are the roots of the quadratic equation (*), then we have $x_1 x_2 = 1$. Thus, if x_1 is a positive real number, so is x_2. The only way for the quadratic equation (*) to have exactly one positive real solution is $x_1 = x_2 = 1$, which implies $k = 4$.

So, $k < 0$ or $k = 4$.

24. Triangle $\triangle ABC$ with sides $AB = 6, BC = 7$, and $AC = 8$ is inscribed in a circle as shown. The point F is on the altitude $\overline{AD}$ and $FD = DX$. Find $\sin(\angle BEA)$.



(a) $\frac{6}{7}$ (b) $\frac{3}{4}$ (c) $\frac{7}{8}$ (d) $\frac{4\sqrt{3}}{7}$ (e) 1

Answer: (e)

Solution: Since $FD = DX$ and $\angle BDF$ is a right angle, the two triangles $\triangle BDX$ and $\triangle BDF$ are congruent. Thus, $\angle DBX$ and $\angle DBF$ have the same measure. Also, note that $\angle DBX$ and $\angle DAC$ have the same measure, half of the measure of the arc $\widehat{XC}$. Thus, the triangles $\triangle BDF$ and $\triangle AEF$ are similar, so the angle $\angle AEF$ is a right angle, and the answer is 1.

25. Let F_n denote the Fibonacci numbers, defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$.

Let S be the sum

$$\sum_{n=1}^{10} \frac{F_n}{5^n} = \frac{F_1}{5} + \frac{F_2}{5^2} + \cdots + \frac{F_{10}}{5^{10}} \quad .$$

Which of the following numbers is closest to S?

(a) $\frac{5}{19}$ (b) $\frac{4}{19}$ (c) $\frac{1}{19}$ (d) $\frac{1}{4}$ (e) $\frac{1}{2}$

Answer: (a)

Solution: Since $S = \sum_{n=1}^{10} \frac{F_n}{5^n}$, then $5S = F_1 + \frac{F_2}{5} + \sum_{n=1}^{8} \frac{F_{n+2}}{5^{n+1}}$. Subtracting S from $5S$, and using the recursive relation $F_{n+2} = F_{n+1} + F_n$, we get $4S \approx F_1 + \frac{F_2}{5} - \frac{F_1}{5} + \frac{S}{5}$, where we dropped the terms $\frac{F_9}{5^9}$ and $\frac{F_{10}}{5^{10}}$, which are smaller than 10^{-4}. Then $S \approx \frac{5}{19}$.

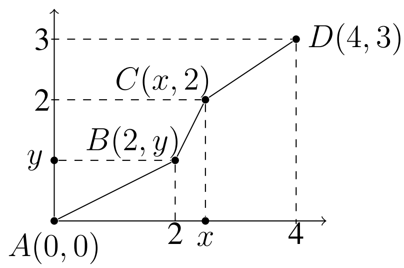
26. Let x and y be real numbers. What is the minimum value of the following function

$$f(x,y) = \sqrt{4+y^2} + \sqrt{(x-2)^2+(2-y)^2} + \sqrt{(4-x)^2+1} \ ?$$

(a) 0 (b) $\frac{7}{2}$ (c) 5 (d) 25 (e) 49

Answer: (c)

Solution: Let $A(0,0)$, $B(2,y)$, $C(x,2)$, and $D(4,3)$ be points in the plane with x and y real numbers.

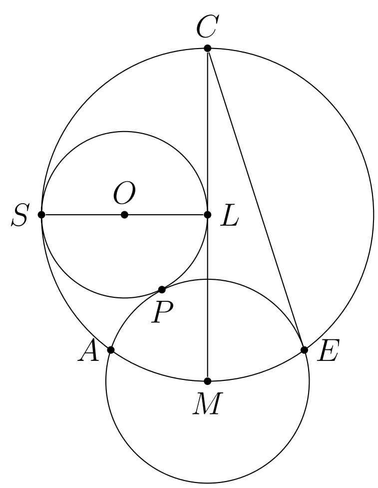


Then, the function $f(x,y)$ is the sum of the distances $AB+BC+CD$. Note that $AB+BC+CD \geq AD$ with equality when B and C are on the line segment $\overline{AD}$ with B between A and C. This can be realized when $x = 8/3$ and $y = 3/2$, so the minimum of $f(x,y)$ is the distance $AD = 5$.

27. Three circles are situated as follows.

The circle SPL has diameter $\overline{SL}$ and it is centered at O; the circle APE is centered at M; and the circle CAE has diameter $\overline{CM}$ and it is centered at L.

The circle CAE is tangent to the circle SPL at S and the circles SPL and APE are tangent at P. Also, $\overline{OL} \perp \overline{CM}$. Find the angle $\angle MCE$.



(a) $15°$ (b) $18°$ (c) $20°$ (d) $24°$ (e) $30°$

Answer: (b)

Solution: Say, the length of LO is 1 (otherwise rescale the whole picture, the angles will not change). Then, $LM = LS = 2LO = 2$.

By the Pythagorean Theorem, $OM = \sqrt{5}$, so $MP = MA = \sqrt{5} - 1$.

Since the arcs $\widehat{AM}$ and $\widehat{ME}$ have the same measure, the measure of $\angle ALM$ (let us denote this measure by α) is twice the measure of $\angle MCE$.

By the Cosine Theorem applied to $\triangle ALM$, we get $\cos(\alpha) = \dfrac{\sqrt{5}+1}{4}$. Using twice that $\cos(2x) = 2\cos^2(x) - 1$, we get $\cos(4\alpha) = -\cos(\alpha)$.

Since both 4α and $\pi - \alpha$ are in $[0, \pi]$ (note that $\dfrac{\sqrt{5}+1}{4} > \dfrac{3}{4} > \dfrac{\sqrt{2}}{2}$), and since the cosine function is strictly decreasing in $[0, \pi]$, we get $4\alpha = \pi - \alpha$, or $\alpha = \pi/5 = 36°$, so the answer is $36°/2 = 18°$.

28. How many different quadruples (x_1, x_2, x_3, x_4) satisfy the equation $x_1 + x_2 + x_3 + x_4 = 18$, where each x_i is a positive integer ≤ 8 ?

(a) 83 (b) 172 (c) 256 (d) 344 (e) 352

Answer: (d)

Solution: First, let us count the number of solutions of the equation

$$(*) x_1 + x_2 + x_3 + x_4 = 18$$

with x_1, x_2, x_3, x_4 positive integers.

Suppose that we pick three distinct integers $a < b < c$ in the interval $[1, 17]$. Then if we set $x_1 = a$, $x_2 = b - a$, $x_3 = c - b$ and $x_4 = 18 - c$, we have that x_1, x_2, x_3, x_4 are positive integers and $x_1 + x_2 + x_3 + x_4 = 18$.

It is not difficult to see that all solutions to $x_1 + x_2 + x_3 + x_4 = 18$ in positive integers can be obtained in this way.

So, the number of solutions of $(*)$ in positive integers is $\dbinom{17}{3} = 680$.

Now, we have the additional restriction that the x's are ≤ 8. We need to count how many solutions have some $x_i \geq 9$ for $i = 1$, or 2, or 3, or 4. Note that at most one of the x's is ≥ 9 (otherwise their sum will be at least $9 + 9 + 1 + 1 > 18$).

Let us count the number of solutions to (*) with $x_4 \geq 9$. Using the same construction as above, to have $x_4 = 18 - c \geq 9$, we need $c \leq 9$.

Thus, we need to pick $a < b < c$ from the interval $[1, 9]$ in this case. There are $\dbinom{9}{3} = 84$ solutions of (*) with $x_4 \geq 9$.

Because of symmetry (any of the x's can be ≥ 9) the number of solutions to exclude is $4 \cdot 84 = 336$.

Thus, the answer is $680 - 336 = 344$.

29. Let x, y, and z be positive real numbers which satisfy the following equations:

$$x^2 + xy + y^2 = 3$$
$$y^2 + yz + z^2 = 1$$
$$z^2 + zx + x^2 = 4.$$

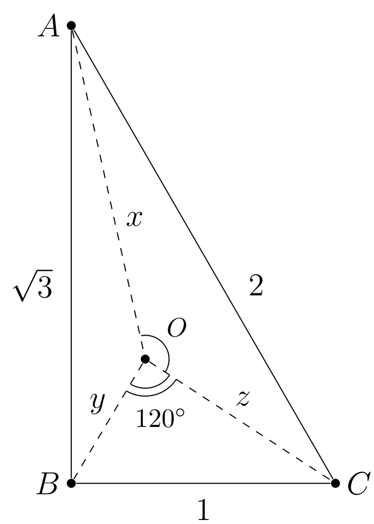
What is the value of the expression $xy + yz + zx$?

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{2}}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) 2 (e) $\frac{2}{\sqrt{3}}$

Answer: (d)

Solution: Note that if $\triangle AOB$ is such that $\angle AOB = 120°$, $AO = x$ and $BO = y$, then by the Cosine Theorem we get $x^2 + xy + y^2 = AB^2$. Suppose x, y, z is a solution in positive real numbers of the equation.

Pick an arbitrary point O, and construct points A, B, and C such that $\angle AOB = \angle BOC = \angle COA = 120°$, $AO = x$, $BO = y$, and $CO = z$.



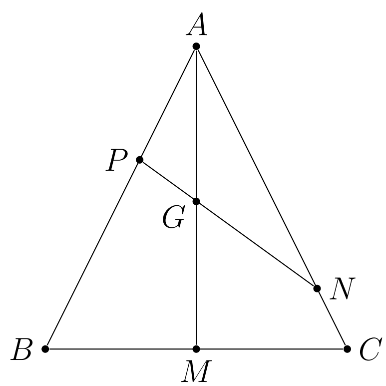
Then, by the Cosine Theorem, $AB = \sqrt{3}$, $BC = 1$, and $CA = 2$. Since $AB^2 + BC^2 = CA^2$, the triangle $\triangle ABC$ is a right triangle and has area $\frac{\sqrt{3}}{2}$.

On the other hand, the area of the triangle $\triangle AOB$ is $\frac{xy \sin(120°)}{2} = \frac{\sqrt{3}}{4}xy$. Similarly, the areas of the triangles $\triangle BOC$ and $\triangle COA$ are $\frac{\sqrt{3}}{4}yz$ and $\frac{\sqrt{3}}{4}xz$, respectively.

Since O is inside the triangle $\triangle ABC$ we get an alternative expression for the area of the triangle $\triangle ABC$: $\frac{\sqrt{3}}{2}(xy + yz + xz)$.

Equating the two expressions for the area of the triangle $\triangle ABC$ we get $xy + xz + yz = 2$.

30. In the triangle $\triangle ABC$, the points P and N are on the sides $\overline{AB}$ and $\overline{CA}$, respectively. Also, $\dfrac{AP}{PB} = \dfrac{1}{2}$ and $\dfrac{CN}{NA} = \dfrac{1}{4}$. What is the ratio $\dfrac{AG}{GM}$ in which the segment $\overline{PN}$ divides the median $\overline{AM}$?



(a) $\dfrac{1}{2}$ (b) $\dfrac{5}{7}$ (c) $\dfrac{7}{8}$ (d) $\dfrac{8}{9}$ (e) $\dfrac{8}{7}$

Answer: (d)

Solution: Denote $\angle BAM = \alpha$, $\angle MAC = \beta$, Area$(\triangle ABC) = S$, Area$(\triangle PAG) = x$, and Area$(\triangle GAN) = y$.

Since $BM = MC$ we have that Area$(\triangle BAM) =$ Area$(\triangle CAM) = S/2$. Also,

$$(*)\ \frac{x}{(S/2)} = \frac{AP \cdot AG \sin(\alpha)/2}{AB \cdot AM \sin(\alpha)/2} = \frac{1}{3} \cdot \frac{AG}{AM}.$$

Similarly,

$$\frac{y}{(S/2)} = \frac{AN \cdot AG \sin(\beta)/2}{AC \cdot AM \sin(\beta)/2} = \frac{4}{5} \cdot \frac{AG}{AM}.$$

Combining the above two equations we get $3x = \dfrac{5y}{4}$, so Area$(\triangle APN) = x + y = \dfrac{17x}{5}$.

Now,

$$\frac{17x/5}{S} = \frac{AP \cdot AN \sin(\alpha+\beta)/2}{AB \cdot AC \sin(\alpha+\beta)/2} = \frac{1}{3} \cdot \frac{4}{5} = \frac{4}{15}.$$

Thus, $\dfrac{x}{S} = \dfrac{4}{51}$. Substituting in (*), we get $\dfrac{AG}{GM} = \dfrac{8}{17}$, or $AG = \dfrac{8}{17}AM$, so $GM = \dfrac{9}{17}AM$.

The answer is $\dfrac{AG}{GM} = \dfrac{8}{9}$.