

High School Math Contest

University of South Carolina

January 30th, 2016

Problem 1. The sum of two prime numbers is 85. What is the product of these two prime numbers?

- (a) 85 (b) 91 (c) 115 (d) 133 (e) 166

Answer: (e)

Solution: Since there is only one even prime number (2), and the sum of two odd numbers is even, the product of these two numbers must be even. \square

Problem 2. A group of people, cats, and birds has seventy legs, thirty heads, and twenty tails. How many cats are among this group? (Assume all birds have two legs and a tail.)

- (a) 0 (b) 5 (c) 10 (d) 15 (e) 20

Answer: (b)

Solution: Write P , C , and B for the number of people, cats, and birds respectively, so that we have

$$2P + 4C + 2B = 70, \quad P + C + B = 30, \quad C + B = 20.$$

Comparing the last two equations we obtain $P = 10$, and the first equation simplifies to $4C + 2B = 50$, or $2C + B = 25$. Therefore,

$$C = (2C + B) - (C + B) = 25 - 5 = 5.$$

Alternatively,

$$C = \frac{1}{2}(2P + 4C + 2B) - (P + C + B) = 35 - 30 = 5. \quad \square$$

Problem 3. A 2-digit number is such that the product of the digits plus the sum of the digits is equal to the number. What is the units digit of the number?

- (a) 1 (b) 3 (c) 5 (d) 7 (e) 9

Answer: (e)

Solution: Assume the 2 digit number is of the form xy , with $x \neq 0$. The statement indicates that these two digits satisfy the equation $xy + x + y = 10x + y$. A quick simplification gives $y = 9$. \square

Problem 4. What is $\log_3(4) \times \log_4(5) \times \cdots \times \log_{80}(81)$?

- (a) 1 (b) 4 (c) 81 (d) $\log_{80} 243$ (e) $e^{81} - 1$

Answer: (b)

Solution: Recall the change of base formula

$$\log_b(a) = \frac{\ln(a)}{\ln(b)}.$$

(Alternatively, we can substitute \log_c for \ln on the right, as long as the base is the same on the top and on the bottom.) Using this, we can rewrite the entire expression using natural logarithms, as

$$\frac{\ln(4)}{\ln(3)} \times \frac{\ln(5)}{\ln(4)} \times \cdots \times \frac{\ln(81)}{\ln(80)} = \frac{\ln(81)}{\ln(3)} = \log_3(81) = \log_3(3^4) = 4. \quad \square$$

Problem 5. You walk one mile to school every day. You leave home at the same time each day, walk at a steady speed of 3 miles per hour, and arrive just as school begins. Today you were distracted by the pleasant weather and walked the first half mile at a speed of only 2 miles per hour. At how many miles per hour must you run the last half mile in order to arrive just as school begins today?

- (a) 4 (b) 6 (c) 8 (d) 10 (e) 12

Answer: (b)

Solution: Recall the formula $s = v \cdot t$, where in this case, s is distance (in miles) traveled at a constant speed of v (miles per hour), for a time of t hours. The first statement indicates that it usually takes you only 20 minutes to go to school (solve for t in $1 = 3 \cdot t$ and change units)

Today, it took you 15 minutes to cover the first half mile (solve for t in $1/2 = 2 \cdot t$ and change units). You only have 5 more minutes to arrive (that is 1/12th of an hour), and need to cover another half mile. Solve for v in the equation $1/2 = (1/12) \cdot v$. \square

Problem 6. An ice cream cone is three inches tall and its top is a circle with diameter two inches. The cone is filled with ice cream, such that the interior of the cone is entirely full. The cone is topped with a hemisphere of ice cream with diameter two inches. What is the total volume, in cubic inches, of the ice cream in and atop the cone?

- (a) π (b) $\frac{4}{3}\pi$ (c) $\frac{3}{2}\pi$ (d) $\frac{5}{3}\pi$ (e) 2π

Answer: (d)

Solution: The volume consists of that of the cone, plus that of the hemisphere on top. The volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$, so with $h = 3$ and $r = \frac{2}{2} = 1$ this volume is $\frac{1}{3}\pi 1^2 \cdot 3 = \pi$. The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$, and we have half this, or $\frac{2}{3}\pi$. The total volume is $\pi + \frac{2}{3}\pi = \frac{5}{3}\pi$. \square

Problem 7. In the Clock Game, part of the game show *The Price Is Right*, a contestant must guess the price (rounded to the nearest dollar) of a prize which is worth less than \$2,000. After each guess, the contestant is told whether her guess was correct, too low, or too high. Assume that the contestant is mathematically savvy but has no idea how much the prize is worth. With how many guesses is she guaranteed to win the prize?

- (a) 10 (b) 11 (c) 12 (d) 13 (e) 1997

Answer: (b)

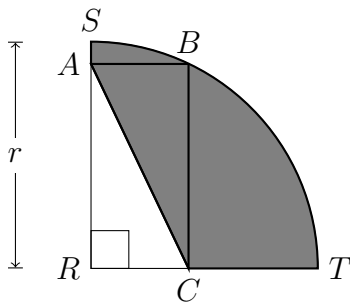
Solution: At each stage, the contestant has narrowed the range of possible prices to an interval, and she should guess exactly in the middle this interval. For example, for the first guess she should guess either \$1,000 or \$1,001. (On the actual show, your *time* is limited rather than the number of guesses, so you should start with \$1,000 because it is quicker to say.)

At every step the contestant can at least halve the number of remaining possibilities. Suppose that the interval has n possible prices and that the contestant's guess does not win immediately. If n is odd there will be $\frac{n-1}{2}$ prices on either side of her guess, and she will be told which of these contains the correct price. If n is even then there will be $\frac{n}{2}$ prices on one side $\frac{n}{2} - 1$ prices on the other, and we assume the worst case of $\frac{n}{2}$.

Since $2^{10} = 1024$ and $\frac{2000}{1024} < 2$, after ten guesses she will be down to one remaining possibility, which she guesses with her eleventh guess.

Conversely, ten guesses is not enough. After each the remaining number of possibilities could be as high as: 1000, 500, 250, 125, 62, 31, 15, 7, 3. She is guaranteed only a 1-in-3 shot in the tenth guess. \square

Problem 8. In the figure below, arc SBT is one quarter of a circle with center R and radius r . The length plus the width of rectangle $ABCR$ is 8, and the perimeter of the shaded region is $10 + 3\pi$. Find the value of r .



- (a) 6 (b) 6.25 (c) 6.5 (d) 6.75 (e) 7

Answer: (a)

Solution: Set $x = RC$ and $y = AR$. It must then be $x + y = 8$ (by hypothesis) and $x^2 + y^2 = r^2$ (since the rectangle $ABCR$ has one vertex in the circle). The perimeter of the shaded area is then $AS + SBT + TC + AC = (r - y) + r\pi/2 + (r - x) + \sqrt{x^2 + y^2} = 3r + r\pi/2 - (x + y) = 3r + r\pi/2 - 8$. All we need to do is solve for r in the equation $18 + 3\pi = (3 + \frac{\pi}{2})r$. \square

Problem 9. One urn contains two pieces of candy—one green and one red. A second urn contains four pieces of candy—one green and three red. For each urn, each piece of candy is equally likely of being picked. You pick a piece of candy from each urn and eat the two chosen candies.

If you eat exactly one piece of green candy, you draw a second piece of candy from the urn still containing a green piece of candy. You now eat the candy you just chose. What is the probability that you ate two pieces of green candy?

- (a) $\frac{1}{8}$ (b) $\frac{1}{4}$ (c) $\frac{3}{8}$ (d) $\frac{1}{2}$ (e) $\frac{5}{8}$

Answer: (c)

Solution: There are four possibilities for the first round:

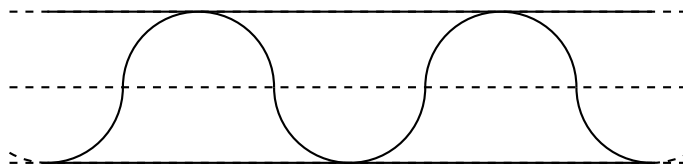
- You pick a green piece of candy from each urn. (Probability $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$.)
- You pick the green piece of candy from the first urn and the red piece of candy from the second. (Probability $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$.) You have a $\frac{1}{3}$ chance of eating the green piece of candy on the second round (since three pieces of candy remain in the second urn, and one of them is green.)
- You pick the red piece of candy from the first urn and the green piece of candy from the second. (Probability $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$.) There is only one more piece of candy in the first urn, so you eat it on the second round.
- You pick a red piece of candy from each urn. (Probability $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$.)

Your chance of eating two pieces of green candy is

$$\frac{1}{8} + \frac{3}{8} \times \frac{1}{3} + \frac{1}{8} = \frac{3}{8}. \quad \square$$

Problem 10. A straight one-mile stretch of highway is 40 feet wide. You ride your bike on a path composed of semicircles as shown. If you ride at 5 miles per hour, how many hours will it take you to cover the one-mile stretch?

Note: 1 mile = 5280 feet



- (a) $\frac{\pi}{11}$ (b) $\frac{\pi}{10}$ (c) $\frac{\pi}{5}$ (d) $\frac{2\pi}{5}$ (e) $\frac{2\pi}{3}$

Answer: (b)

Solution: We wish to solve for t in the equation $s = v \cdot t$, where s is the distance covered in feet, and $v = 5 \cdot 5280$ feet an hour.

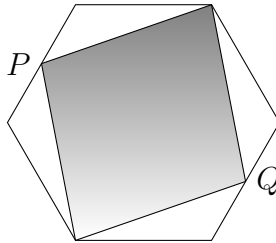
Note that each semicircle has a radius of 20 feet (half the width of the road). This means that you must complete $5280/40$ semicircles to cover the one-mile stretch. Each of those

semicircles measures 20π feet. You ride then a grand total of $\frac{5280}{40} \times 20\pi = \frac{5280}{2}\pi$ feet. We have then

$$\frac{5280}{2}\pi = 5 \cdot 5280 \cdot t,$$

which gives $t = \frac{\pi}{10}$ hours. □

Problem 11. In the diagram below, the points P and Q are the midpoints of opposite sides of a regular hexagon. What fraction of the hexagon is shaded?



- (a) $2/3$ (b) $3/4$ (c) $5/6$ (d) $7/8$ (e) $11/12$

Answer: (a)

Solution: Assume the hexagon has side length b . The total area of the hexagon is then 6 times the area of a equilateral triangle side-length b ; this is, $6bh/2 = 3b^2\sqrt{3}$. The area of the shaded quadrilateral can be seen as the addition of the area of two isosceles triangles with base $2b$ and height $\sqrt{3}b/2$. The requested fraction is then $2/3$. □

Problem 12. One day in December 2015, three 2-digit prime numbers A , B and C , were given to three members of a High School math team: Ashley, Beth, and Caitlin (respectively). They had this conversation:

Ashley: “If you two add your numbers, we get precisely today’s date!”

Beth: “If you two add your numbers, we get my birthday this month, which was before today.”

Caitlin: “If you two add your numbers, we get my birthday this month, which is after today.”

What number did Caitlin get?

- (a) 11 (b) 13 (c) 17 (d) 19 (e) 23

Answer: (a)

Solution: The numbers A , B , C belong a priori in the set $\{11, 13, 17, 19\}$ —Note that the addition of any of these two 2-digit prime numbers cannot be greater than or equal to 30, which places a strong restriction on the greatest of the three numbers: this must be at most 19. These are the only valid additions of any two numbers on that set:

$$11 + 11 = 22, \quad 11 + 13 = 24, \quad 11 + 17 = 29, \quad 11 + 19 = 30, \quad 13 + 13 = 26, \quad 13 + 17 = 30$$

Note in particular the strict inequalities placed on the addition of any of these two numbers:

$$22 \leq A + C < B + C < A + B \leq 30$$

Since $C < B$, there are only two possibilities for $B + C$ at this point: $B = 13, C = 11, B + C = 24$, or $B = 17, C = 11, B + C = 29$. □

Problem 13. Every day, the value of a stock rises by exactly two dollars in the morning, and then falls by exactly one dollar in the afternoon. If the stock's value at the start of the first day is \$100, on what day will the stock's value first reach \$200?

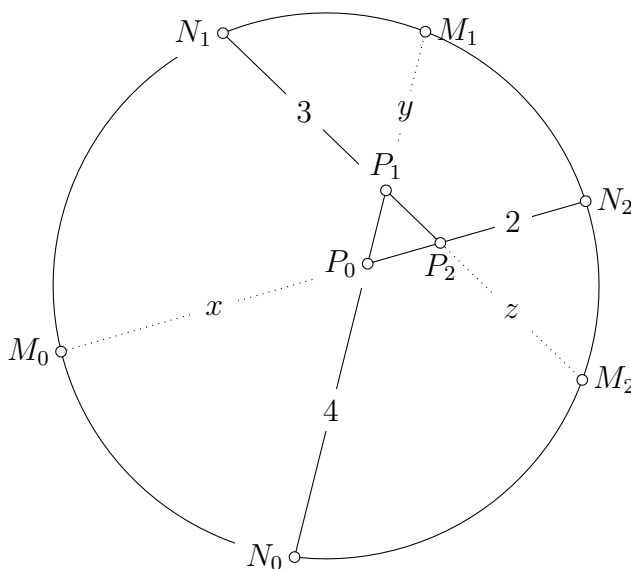
- (a) 50 (b) 99 (c) 100 (d) 101 (e) 200

Answer: (b)

Solution: Note that each day the stock is worth \$1 more than on the previous day, and that on the first day it rises to \$102 and then falls to \$101. Therefore, on the n th day, the stock's value rises to $n + 101$ dollars in the morning and falls to $n + 100$ dollars in the afternoon.

The solution is therefore the smallest n for which $n + 101 = 200$, or $n = 99$. In particular, the stock will be \$200 in the middle of day 99, although not at the end of this day. \square

Problem 14. In the diagram below, a triangle $\triangle P_0P_1P_2$ is equilateral with side-length 1. The length of segment $\overline{P_0N_0}$ is 4, the length of segment $\overline{P_1N_1}$ is 3, and the length of segment $\overline{P_2N_2}$ is 2. Let $x = P_0M_0$, $y = P_1M_1$, and $z = P_2M_2$. Compute $x + y + z$.



- (a) 6 (b) 7 (c) 8 (d) 9 (e) 10

Answer: (d)

Solution: An application of the Intersecting Chords Theorem for the chords $\overline{N_0M_1}$ and $\overline{M_0N_2}$ gives $4(1 + y) = 3x$. Similarly, for the chords $\overline{N_0M_1}$ and $\overline{N_1M_2}$, it is $3(1 + z) = 5y$. Finally, for the chords $\overline{M_0N_2}$ and $\overline{N_1M_2}$, we have $2(1 + x) = 4z$. Adding these three equations together, we obtain $2 + 2x + 4 + 4y + 3 + 3z = 3x + 5y + 4z$, which simplifies to $x + y + z = 9$. \square

Problem 15. Seven different playing cards, with values from ace to seven, are shuffled and placed in a row on a table to form a seven-digit number. What is the probability that this seven-digit number is divisible by 11?

Note: Each of the possible seven-digit numbers is equally likely to occur.

- (a) 4/35 (b) 1/7 (c) 8/35 (d) 2/7 (e) 12/35

Answer: (a)

Solution: To be divisible by 11, the digits must be arranged so that the difference between the sum of one set of alternate digits, and the sum of the other set of alternate digits, is either 0 or a multiple of 11. The sum of all seven digits is 28. It is easy to find that 28 can be partitioned in only two ways that meet the 11 test: 14|14, and 25|3. The 25|3 is ruled out because no sum of three different digits can be as low as 3. Therefore, only the 14|14 partition need to be considered.

There are 35 different combinations of three digits that can fall into the B positions in the number ABABABA. Of those 35, only four sum to 14: 167, 257, 347, and 356. Therefore, the probability that the number will be divisible by 11 is $4/35$. \square

Problem 16. Given the sequence $\{x_n\}$ defined by $x_{n+1} = \frac{1+x_n\sqrt{3}}{\sqrt{3}-x_n}$ with $x_1 = 1$, compute the value of $x_{2016} - x_{618}$.

- (a) 0 (b) 1 (c) $\sqrt{3}$ (d) 2 (e) $2\sqrt{3}$

Answer: (a)

Solution: Note that $x_2 = 2 + \sqrt{3}$, $x_3 = -2 - \sqrt{3}$, $x_4 = -1$, $x_5 = -2 + \sqrt{3}$, $x_6 = 2 - \sqrt{3}$, $x_7 = 1 = x_1$, so that we have $x_{n+6} = x_n$ for all positive integers n ; i.e., the sequence is periodic. Since $2016 - 618 = 1398 = 233 \times 6$, we have $x_{2016} - x_{618} = 0$. \square

Problem 17. Compute the minimum value of the expression $\sin^{2016} \alpha + \cos^{2016} \alpha$, for $\alpha \in \mathbb{R}$.

- (a) 2^{-1007} (b) 2^{-1006} (c) 2^{-1005} (d) 2^{-1004} (e) 2^{-1003}

Answer: (a)

Solution: Set $x = \sin^2 \alpha$, so we can re-write the given expression as the polynomial $x^{1008} + (1-x)^{1008}$. We are then looking for the minimum of this expression for $0 \leq x \leq 1$. Set $F(x)$ to be the previous polynomial. Note that $F(x) = F(1-x)$, so there is an obvious symmetry about $x = 1/2$. Note also that $F(0) = F(1) = 1$, and $F(1/2) = 2/2^{1008} = 2^{-1007}$. Make again a change of variable to shift the center to zero: let $y = x - \frac{1}{2}$. It is then $F(y) = (y + \frac{1}{2})^{1008} + (y - \frac{1}{2})^{1008}$. Note that, written this way, we can see that the odd powers of y cancel, so that $F(y)$ is the sum of 2^{-1007} and even powers of y with positive coefficients. In particular $F(y)$ is always at least 2^{-1007} , so that this value is the minimum. \square

Problem 18. Suppose $f(x)$ is an odd function for which $f(x+2) = f(x)$ for all x , and $f(x) = x^2$ for $x \in (0, 1)$. Compute $f(-3/2) + f(1)$.

- (a) -1 (b) $-\frac{1}{2}$ (c) $-\frac{1}{4}$ (d) $\frac{1}{4}$ (e) $\frac{1}{2}$

Answer: (d)

Solution: Because f is periodic, we know that $f(-3/2) = f(1/2) = (1/2)^2 = 1/4$. Because f is odd, we know that $f(1) = -f(-1)$, but because f is periodic, $f(1) = f(-1)$. Therefore, $f(1) = 0$ and the answer is $1/4$. \square

Problem 19. Consider the sequence $\{a_n\}$ given by $a_{n+1} = \arctan(\sec a_n)$, with $a_1 = \frac{\pi}{6}$. Find the value of a positive integer m that satisfies $\sin a_1 \sin a_2 \cdots \sin a_m = \frac{1}{100}$.

- (a) 333 (b) 334 (c) 666 (d) 3333 (e) 6666

Answer: (d)

Solution: From the given conditions, we have $a_{n+1} \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and $\tan a_{n+1} = \sec a_n$. Since $\sec a_n > 0$, we have $a_{n+1} \in (0, \frac{\pi}{2})$ and furthermore, $\tan^2 a_{n+1} = \sec^2 a_n = 1 + \tan^2 a_n$. Therefore,

$$\tan^2 a_n = n - 1 + \tan^2 a_1 = n - 1 + \frac{1}{3} = \frac{3n - 2}{3}.$$

Then

$$\begin{aligned} \sin a_1 \sin a_2 \cdots \sin a_m &= \frac{\tan a_1}{\sec a_1} \frac{\tan a_2}{\sec a_2} \cdots \frac{\tan a_m}{\sec a_m} \\ &= \frac{\tan a_1}{\tan a_2} \frac{\tan a_2}{\tan a_3} \cdots \frac{\tan a_m}{\tan a_{m+1}} \\ &= \frac{\tan a_1}{\tan a_{m+1}} = \sqrt{\frac{1}{3m+1}}. \end{aligned}$$

Solving for m in the equation

$$\sqrt{\frac{1}{3m+1}} = \frac{1}{100},$$

we obtain $m = 3333$. □

Problem 20. The polynomial $x^4 - 27x^2 + 121$ can be factored in a unique way into a product of two quadratic polynomials with integer coefficients and leading coefficient 1. What is the sum of these two polynomials?

- (a) $2x^2 - 5x + 122$ (b) $2x^2 - 5x - 22$ (c) $2x^2 - 5x + 22$ (d) $2x^2 - 22$ (e) $2x^2 + 22$

Answer: (e)

Solution: Write $x^4 - 27x^2 + 121 = (x^2 + ax + b)(x^2 + cx + d)$. Since the x^3 and x coefficients are zero, we have $c = -a$ and $ad = -bc$, hence $ad = ab$. We also have $bd = 121$, so that each of b, d is $\pm 1, \pm 11$, or ± 121 .

We check that $a \neq 0$: if we had $a = c = 0$, then we would have $b + d = -27$, impossible with any of the values of b and d listed earlier. Therefore $b = d$, and b and d must both be either 11 or -11 .

We thus have

$$x^4 - 27x^2 + 121 = (x^2 + ax + b)(x^2 - ax + b).$$

Looking at the x^2 coefficient, we have that $-27 = -a^2 + 2b$. Since $b = \pm 11$, we have $-a^2$ equal to -5 or -49 , hence -49 (since -5 is not a perfect square). So $2b = 22$, $b = 11$, and $a = \pm 7$. We conclude that our desired factorization is

$$x^4 - 27x^2 + 121 = (x^2 + 7x + 11)(x^2 - 7x + 11). \quad \square$$

Problem 21. You choose at random ten points inside of a circle, so that no two of them are on any diameter. What is the probability that the circle has some diameter, so that exactly five points are on one side and exactly five points are on the other?

- (a) $\frac{63}{256}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{5}{6}$ (e) 1

Answer: (e)

Solution: The randomness is a red herring: There is some diameter dividing the points 5 – 5 no matter what!

Note that none of the points can be in the center of the circle. (Then any diameter going through any of the points would contain two points.) Choose any diameter. If five points are on either side, we're done. Otherwise, let n be the larger number of points and rotate the diameter gradually 180 degrees. As we do this, the number of points on the side with n changes whenever the diameter crosses a point, eventually dropping to $10 - n$. Since 5 is between n and $10 - n$, and this number of points changes one at a time, it must be 5 at some point. \square

Problem 22. Let x_1, x_2, \dots, x_k be the distinct real solutions of the equation $x^3 - 3[x] = 4$. Compute $x_1^3 + x_2^3 + \dots + x_k^3$.

- (a) 13 (b) 14 (c) 15 (d) 16 (e) 17

Answer: (c)

Solution: For convenience, write $f(x) = x^3 - 3[x] - 4$. Now $f(-3) = -22$ (and $f(x) \approx -19$ for x very slightly smaller than -3) and $f(3) = 14$. Since $f(x)$ is approximately increasing for $x > 3$ and $x < -3$, we can reliably guess that we cannot have $f(x) = 0$ in these regions, and it is not too much additional work to prove this.

We consider the possibility of solutions in each interval $[n, n + 1)$ for each integer n with $-3 \leq n \leq 2$:

- $n = -3$, $f(x) = x^3 + 5$. No solution because $\sqrt[3]{-5} \notin [-3, -2)$.
- $n = -2$, $f(x) = x^3 + 2$. One solution $\sqrt[3]{-2}$.
- $n = -1$, $f(x) = x^3 - 1$. No solution because $1 \notin [-1, 0)$.
- $n = 0$, $f(x) = x^3 - 4$. No solution because $\sqrt[3]{4} \notin [0, 1)$.
- $n = 1$, $f(x) = x^3 - 7$. One solution $\sqrt[3]{7}$.
- $n = 2$, $f(x) = x^3 - 10$. One solution $\sqrt[3]{10}$.

The answer is $-2 + 7 + 10 = 15$. \square

Problem 23. Let $\alpha_1, \dots, \alpha_k$ be the distinct real numbers in $[0, 2\pi]$ which satisfy the equation $\sin x + \cos x = -1/3$. What is the value of $\alpha_1 + \dots + \alpha_k$?

- (a) $7\pi/4$ (b) 2π (c) $9\pi/4$ (d) $5\pi/2$ (e) $11\pi/4$

Answer: (d)

Solution: Note that the given equation is equivalent to $\sin(x + (\pi/4)) = -1/(3\sqrt{2})$, which has exactly two solutions in $[0, 2\pi]$. Observe that

- (a) $x > \pi/2$ (since $\sin x + \cos x < 0$), and
- (b) x satisfies the given equation if and only if $(\pi/2) - x$ does.

Thus, if α is a solution in $[0, 2\pi]$, then the other solution is $(5\pi/2) - \alpha$ (where the possibility that $\alpha = (5\pi/2) - \alpha$ is easily eliminated). Hence, the answer is $5\pi/2$. \square

Problem 24. If n is a positive integer, write $s(n)$ for the sum of the digits of n . What is $s(1) + s(2) + s(3) + \dots + s(1000)$?

- (a) 9991
- (b) 10000
- (c) 13501
- (d) 14999
- (e) 15000

Answer: (c)

Solution: We start by computing $S := s(0) + s(1) + s(2) + s(3) + \dots + s(999)$. Each digit, 0 through 9, appears equally often: 100 times in the hundreds place, 100 times in the tens place, and 100 times in the ones place. (Here we write, e.g., 3 as 003.) Therefore,

$$S = 300 \cdot 0 + 300 \cdot 1 + \dots + 300 \cdot 9 = 300(0 + 1 + \dots + 9) = 300 \cdot 45 = 13500.$$

Since $s(0) = 0$ and $s(1000) = 1$ we obtain the answer. \square

Problem 25. Consider an infinite supply of cardboard equilateral triangles and squares, all of them with side length of one inch. What would be the convex polygon (without holes) with the largest number of sides that you could construct with these pieces, if the pieces are not allowed to overlap?

- (a) hexagon
- (b) octagon
- (c) 10-gon
- (d) 12-gon
- (e) 16-gon

Answer: (d)

Solution: Recall first that the interior angles of a convex polygon with n sides add up to $180n - 360$ (in degrees).

By construction, all the convex angles at the proper exterior vertices of our constructed polygons can only be 60, 90, 120, or 150 degrees.

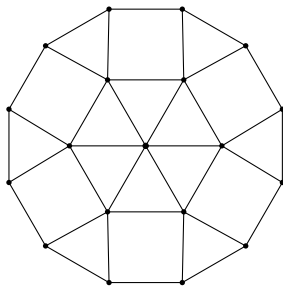
Let a, b, c be the number of exterior vertices of such a polygon with an convex angle of 60, 90 or 120 degrees, respectively (there are thus $n - a - b - c$ vertices presenting convex angles of 150 degrees). These variables verify then the following equation:

$$\underbrace{60a + 90b + 120c + 150(n - a - b - c)}_{\text{add all degrees of each vertex}} = 180n - 360.$$

This identity reduces, after simplification, to $3a + 2b + c + n = 12$.

This is a very strong statement! It indicates, among other things, that such polygons cannot have more than 12 sides. Take for example the case of the 12-sided polygon. According to the previous constraint, it can only be $a = 0, b = 0, c = 0$, which indicates that this polygon

can only have exterior vertices with convex angle 150 degrees. Such a construction is possible, and one example is shown below:



□

Problem 26. Two positive numbers a and b satisfy $2 + \log_2 a = 3 + \log_3 b = \log_6(a + b)$. Compute the value of $\frac{1}{a} + \frac{1}{b}$.

- (a) 2 (b) 3 (c) 108 (d) 216 (e) 324

Answer: (c)

Solution: Suppose $k = 2 + \log_2 a = 3 + \log_3 b = \log_6(a + b)$, then we have $a = 2^{k-2}$, $b = 3^{k-3}$, $a + b = 6^k$, and therefore, $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{6^k}{2^{k-2}3^{k-3}} = 2^2 \times 3^3 = 108$. □

Problem 27. Consider the sequence $\{a_n\}$ given by $a_{n+1} = 2(n+2)a_n/(n+1)$, with $a_1 = 2$. Compute the value of

$$\frac{a_{2016}}{a_1 + a_2 + \cdots + a_{2015}}.$$

- (a) $\frac{2016}{2015}$ (b) $\frac{2016}{2014}$ (c) $\frac{2017}{2016}$ (d) $\frac{2017}{2015}$ (e) $\frac{2017}{2014}$

Answer: (d)

Solution: From the definition, we have

$$\begin{aligned} a_n &= \frac{2(n+1)}{n} a_{n-1} = \frac{2(n+1)}{n} \cdot \frac{2n}{n-1} a_{n-2} = \cdots \\ &= \frac{2(n+1)}{n} \cdot \frac{2n}{n-1} \cdots \frac{2 \cdot 3}{2} a_1 = 2^{n-1}(n+1) \end{aligned}$$

The sum of the first n terms of $\{a_n\}$, denoted by S_n , is given by:

$$S_n = 2 + 2 \times 3 + 2^2 \times 4 + \cdots + 2^{n-1}(n+1).$$

Then

$$2S_n = 2^2 + 2^2 \times 3 + 2^3 \times 4 + \cdots + 2^n(n+1),$$

so the difference of the above two equations gives

$$\begin{aligned} S_n &= 2^n(n+1) - (2 + 2 + 2^2 + \cdots + 2^{n-2} + 2^{n-1}) \\ &= 2^n(n+1) - 2^n = 2^n n. \end{aligned}$$

Therefore,

$$\frac{a_{2016}}{a_1 + a_2 + \cdots + a_{2015}} = \frac{2^{2015} \times 2017}{2^{2015} \times 2015} = \frac{2017}{2015}.$$

□

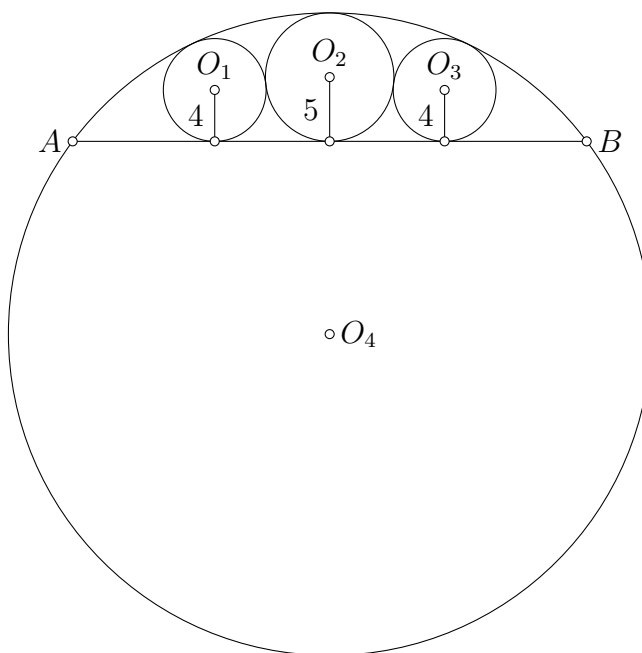
Problem 28. Let $x = 0.82^{0.5}$, $y = \sin 1$ (radians), and $z = \log_3 \sqrt{7}$. Which of the following statements is true?

- (a) $x < y < z$ (b) $y < z < x$ (c) $y < x < z$ (d) $z < y < x$ (e) $z < x < y$

Answer: (b)

Solution: Since 1 radian is less than 60 degrees, we have $y = \sin(1) < \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} < \frac{7}{8}$. Since $3^7 < 7^4$, we take \log_3 of both sides, then we have $7 < \log_3 7^4 = 4 \log_3 7$ which gives $\frac{7}{8} < \log_3 \sqrt{7} = z$, so $y < z$; on the other hand, since $3^9 > 7^5$, so $9 > \log_3 7^5 = 5 \log_3 7$, then $z = \log_3(\sqrt{7}) < \frac{9}{10} = 0.9$. Since $x = \sqrt{0.82} > \sqrt{0.81} = 0.9$, so we have $y < z < x$. \square

Problem 29. The figure below shows four tangent circles: C_1 with center O_1 and radius 4, C_2 with center O_2 and radius 5, C_3 with center O_3 and radius 4, and C_4 with center O_4 . Chord \overline{AB} of circle C_4 is tangent to circles C_1 , C_2 and C_3 . Find AB .



- (a) 30 (b) 32 (c) 35 (d) 37 (e) 40

Answer: (e)

Solution: Let $\overline{O_1D}$ be an altitude of the triangle $\triangle O_1O_2O_4$ and let $O_1D = h$. From triangle $\triangle O_1O_2D$, we have $h^2 = 9^2 - 1 = 80$. If the radius of C_4 is r , then from triangle $\triangle O_1O_4D$ we obtain $(r - 4)^2 = h^2 + (r - 6)^2$. Simplifying and using the value we found for h , we obtain $r = 25$.

An application of the Intersecting Chords Theorem in circle C_4 with the chord \overline{AB} and the chord that goes through O_4 and O_2 indicates that we must have $(\frac{AB}{2})^2 = 10(2r - 10) = 400$. This resolves in $AB = 40$. \square

Problem 30. Find $\arctan(1/3) + \arctan(1/5) + \arctan(1/7) + \arctan(1/8)$.

- (a) $\pi/6$ (b) $\pi/5$ (c) $\pi/4$ (d) $\pi/3$ (e) $\pi/2$

Answer: (c)

Solution: One solution can be given in terms of the angle addition formula for tangent. Recall that

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}.$$

If we write $u = \tan \alpha$ and $v = \tan \beta$, then by taking arctangent of both sides this can be rewritten as

$$\arctan(u) + \arctan(v) = \arctan\left(\frac{u + v}{1 - uv}\right).$$

So, for example, we have

$$\arctan(1/3) + \arctan(1/5) = \arctan\left(\frac{1/3 + 1/5}{14/15}\right) = \arctan(4/7).$$

Applying the identity twice more we compute that the answer is $\arctan(1)$, or $\pi/4$.

A second, equivalent, solution uses complex numbers. If $z = a + bi = re^{i\theta}$ is a complex number with a and b positive, then $\theta = \arctan(b/a)$. Moreover, if $z_1 = a_1 + b_1i = r_1e^{i\theta_1}$ and $z_2 = a_2 + b_2i = r_2e^{i\theta_2}$, then we have $z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$ – i.e., multiplication of complex numbers corresponds to addition of the angles.

Therefore, if we choose four complex numbers whose angles are the arctangents above, e.g.

$$z_1 = 3 + i, \quad z_2 = 5 + i, \quad z_3 = 7 + i, \quad z_4 = 8 + i,$$

the solution is the angle of $z_1z_2z_3z_4$. A computation shows that $z_1z_2z_3z_4 = 650 + 650i$, which has angle $\pi/4$. □